

On Narayana numbers which are products of four b -repdigits with a consequence

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Received 20 November 2025; Revised 4 December 2025; Accepted 31 December 2025

ABSTRACT. In this paper, we focus on Narayana numbers which can be written as a products of four repdigits in base g , where g is an integer with $g \geq 2$. We prove that for g between 2 and 12, there are finitely many of these numbers. Moreover we have fully determined them.

2020 AMS Classification: 11B39, 11J86, 11D61, 11Y50

Keywords: Diophantine equations, Narayana numbers, g -repdigits, Linear forms in logarithms, Reduction method.

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1. INTRODUCTION

In this paper we study an exponential Diophantine equation. A Diophantine equation is a polynomial equation, usually with two or more unknowns, for which only integer solutions are sought. They can be linear (sum of two or more monomials of degree 1) or exponential (the exponents on the terms can be unknown). We will be specifically interested in the Narayana numbers which can be written as products of four repdigits in base g with $g \geq 2$.

The Narayana sequence was introduced by the Indian mathematician Narayana Pandita in the 14th century in his treatise *Ganita Kaumudi*. In this book, he poses a problem about the growth of cows, very similar to Fibonacci problem, but with a different maturity period. In this problem, it is assumed that a calf is born each year from every cow that is at least three years old. Narayana's cow's problem counts the number of calves produced each year [1]. The *Narayana sequence* is define by:

$$N_n = N_{n-1} + N_{n-3} \text{ for } n \geq 3 \text{ with } N_0 = 0, N_1 = N_2 = 1.$$

In fixed base $g \geq 2$, a repdigit has the following form:

$$\sum_{i=0}^{n-1} d \times g^i = d \times \frac{g^n - 1}{g - 1}, \text{ where } 1 \leq d \leq g - 1.$$

When $g = 10$, one usually omits to mention g and simply call these numbers as repdigits.

Several studies have been conducted on determining the terms of linear recurrence sequences that are repdigits in any base $g \geq 2$. For more details refer to the recent results [2, 3, 4, 5, 6, 7, 8, 9, 10]. on double application of Baker’s method and a reduction algorithm using computations based on continued fractions. In this paper, we will use the same approach for identifying Narayana numbers that are products of four repdigits in base g .

The present paper is organized as follows. In Section 2 we present the main results; Section 3 recalls the necessary preliminaries; Section 4 contains the proofs.

2. STATEMENT OF MAIN RESULTS

In this section, we state all the main results obtained in this paper.

Theorem 2.1. *Let $g \geq 2$ be an integer. Then the Diophantine equation*

$$(2.1) \quad N_k = d_1 \frac{g^\ell - 1}{g - 1} \cdot d_2 \frac{g^m - 1}{g - 1} \cdot d_3 \frac{g^n - 1}{g - 1} \cdot d_4 \frac{g^t - 1}{g - 1}$$

has only finitely many solutions in integers $k, d_1, d_2, d_3, d_4, \ell, m, n, t$ such as

$$1 \leq d_i \leq g - 1 \text{ for } i = 1, 2, 3, 4 \text{ and } 1 \leq \ell \leq m \leq n \leq t.$$

Furthermore, we have

$$t < 2.11 \times 10^{67} \log^{12} g \text{ and } k < 2.54 \times 10^{68} \log^{13} g.$$

In the following theorem, we completely and explicitly give all solutions of equation (2.1) corresponding to $2 \leq g \leq 12$.

Theorem 2.2. $\{1, 2, 3, 4, 6, 9, 13, 28, 60, 88, 129, 189\}$ *is the set of only Narayana numbers which are products of four repdigits in base g with $2 \leq g \leq 12$.*

Considering the notions introduced in Theorem 2.1, the solutions to the equation (2.1) can be expressed in the form:

$$N_k = [a, b, c, d]_g = a \times b \times c \times d \text{ in base } g,$$

where

$$a = d_1 \frac{g^\ell - 1}{g - 1}, b = d_2 \frac{g^m - 1}{g - 1}, c = d_3 \frac{g^n - 1}{g - 1}, d = d_4 \frac{g^t - 1}{g - 1}.$$

Then we have:

TABLE 1. Narayana numbers that are products of four repdigits in base g for $2 \leq g \leq 12$.

k	N_k	$[a, b, c, d]_g$
1, 2, 3	1	$[1, 1, 1, 1]_g$ for $g = 2, \dots, 12$
4	2	$[1, 1, 1, 2]_g$ for $g = 3, \dots, 12$
5	3	$[1, 1, 1, 11]_2, [1, 1, 1, 3]_g$ for $g = 4, \dots, 12$
6	4	$[1, 1, 1, 11]_3, [1, 1, 1, 3]_g$ for $g = 5, \dots, 12$.
7	6	$[1, 1, 1, 11]_5, [1, 1, 2, 3]_g$ for $g = 4, \dots, 12, [1, 1, 1, 6]_g$ for $g = 7, \dots, 12$
8	9	$[1, 1, 11, 11]_2, [1, 1, 1, 11]_8, [1, 1, 1, 9]_{10}, [1, 1, 3, 3]_g$ for $g = 4, \dots, 12$
9	13	$[1, 1, 1, 111]_3, [1, 1, 1, 11]_{12}$
11	28	$[1, 1, 1, 44]_6, [1, 1, 2, 22]_6, [1, 1, 4, 11]_6, [1, 1, 4, 7]_g$ for $g = 8, \dots, 12, [1, 2, 2, 7]_g$, for $g = 8, \dots, 12$
13	60	$[1, 2, 2, 33]_4, [1, 2, 3, 22]_4, [2, 2, 3, 5]_4, [1, 1, 1, 66]_9, [1, 1, 2, 33]_9, [1, 1, 3, 22]_9, [1, 1, 6, 11]_9, [1, 2, 3, 11]_9, [1, 1, 1, 55]_{11}, [1, 1, 5, 11]_{11}, [1, 1, 6, A]_g, [1, 2, 3, A]_g$ for $g = 11, 12, [1, 2, 5, 6]_g$ for $g = 7, \dots, 12, [1, 3, 4, 5]_g, [2, 2, 3, 5]_g$ for $g = 6, \dots, 12$
14	88	$[1, 1, 1, 88]_{10}, [1, 1, 8, 11]_g, [1, 2, 4, 11]_g, [2, 2, 2, 11]_g$ for $g = 10, 12, [1, 1, 2, 44]_{10}, [1, 1, 4, 22]_{10}, [1, 2, 2, 22]_{10}$
15	129	$[1, 1, 1, 333]_6, [1, 1, 3, 111]_6$
16	189	$[1, 1, 11, 111111]_2, [11, 11, 11, 111]_2, [1, 1, 3, 333]_4, [1, 3, 3, 111]_4, [1, 3, 3, 33]_6, [3, 3, 3, 11]_6, [1, 1, 3, 77]_8, [1, 1, 7, 33]_8, [3, 3, 3, 7]_g$ for $g = 8, \dots, 12, [1, 3, 7, 9]_g$ for $g = 8, 10, 11, 12$

3. PRELIMINARIES

In this section, we will cover the essential concepts needed for the proofs of the results stated in section 2.

3.1. Some properties of Narayana sequence.

Definition 3.1. The Narayana sequence $(N_n)_{n \geq 0}$ is defined by the third-order linear recurrence relation

$$N_n = N_{n-1} + N_{n-3}, \quad n \geq 3,$$

with initial conditions $N_0 = 0$ and $N_1 = N_2 = 1$. Its characteristic equation is

$$x^3 - x^2 - 1 = 0.$$

The characteristic polynomial $f(x) = x^3 - x^2 - 1$ has three roots α, β and γ , given by

$$\begin{aligned} \alpha &= \frac{1}{3} \left(\sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} + \sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})} + 1 \right), \\ \beta &= \frac{1}{3} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} - \frac{1}{6}(1 + i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}, \\ \gamma &= \frac{1}{3} - \frac{1}{6}(1 + i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 - 3\sqrt{93})} - \frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{\frac{1}{2}(29 + 3\sqrt{93})}. \end{aligned}$$

Noticing that it has a real zero $\alpha (> 1)$ and two conjugate complex roots β and γ with $|\beta| = |\gamma| < 1$. We have $\alpha \approx 1.46557$.

Lemma 3.2. *Let $(N_n)_{n \geq 0}$ be a sequence.*

(1) *we have*

$$(3.1) \quad \alpha^{n-2} \leq N_n \leq \alpha^{n-1} \text{ for all } n \geq 1.$$

(2) *$(N_n)_{n \geq 0}$ satisfies the following "Binet-like" formula*

$$(3.2) \quad N_n = a\alpha^n + b\beta^n + c\gamma^n \text{ for all } n \geq 0,$$

(3) *from the three initial values of Narayana sequence, and using Vieta's theorem, one has:*

$$a = \frac{\alpha^2}{\alpha^3 + 2}, \quad b = \frac{\beta^2}{\beta^3 + 2}, \quad \text{and } c = \frac{\gamma^2}{\gamma^3 + 2}.$$

If we write $\zeta_n = N_n - a\alpha^n = b\beta^n + b\gamma^n$, then

$$(3.3) \quad |\zeta_n| < \frac{1}{\alpha^{n/2}} \text{ for all } n \geq 1.$$

Proof. By induction on n , we simply prove (1). The proof of (2) can be found in reference [11]. Finally, the proof of (3) follows from the triangle inequality and the fact that $|\beta| = |\gamma| < 1$. We leave the details to the reader. \square

Let $\mathbb{K}_f := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial f over \mathbb{Q} . Then $[\mathbb{K}_f : \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of \mathbb{K}_f over \mathbb{Q} is given by

$$\begin{aligned} \mathcal{G}_f := Gal(\mathbb{K}_f/\mathbb{Q}) &\cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \\ &\cong S_3. \end{aligned}$$

Thus we identify the automorphisms of \mathcal{G}_f with the permutation of the zeros of the polynomial f . For example, the permutation $(\alpha\beta)$ corresponds to the automorphisms $\sigma_f : \alpha \rightarrow \beta, \beta \rightarrow \alpha, \gamma \rightarrow \gamma$.

Recall that α is the real root > 1 of the characteristic equation $\alpha^3 - \alpha^2 - 1 = 0$. From the expression $a = \frac{\alpha^2}{\alpha^3 + 2}$, and using $\alpha^3 = \alpha^2 + 1$, we obtain the simplified form $a = \frac{\alpha^2}{\alpha^2 + 3}$. Let $s = \alpha^2$. Then $a = \frac{s}{s + 3}$. Thus $s = \frac{3a}{1 - a}$. On the other hand, since $\alpha^2 = s$ and $\alpha^3 = \alpha s$, the equation $\alpha^3 - \alpha^2 - 1 = 0$ becomes $\alpha s - s - 1 = 0$, which implies $\alpha = \frac{s + 1}{s}$. Substituting this expression into $\alpha^2 = s$ yields $s = \left(\frac{s + 1}{s}\right)^2$, which

is equivalent to $s^3 - s^2 - 2s - 1 = 0$. Replacing $s = \frac{3a}{1-a}$ in this cubic equation and clearing denominators gives the polynomial $31a^6 - 93a^5 + 90a^4 - 23a^3 - 6a^2 + 1 = 0$. This polynomial factors over \mathbb{Q} as $(1-a)^3(31a^3 - 3a - 1) = 0$. Since $0 < a < 1$, the factor $(1-a)^3$ is extraneous. So the minimal polynomial of a over \mathbb{Q} is $31x^3 - 3x - 1$.

3.2. Linear form in logarithms. This subsection opens with a brief overview of key facts about the logarithmic height of algebraic numbers.

For an algebraic number η of degree d over \mathbb{Q} and minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^d (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

with positive leading coefficient a_0 , we write $h(\eta)$ for its logarithmic height, given by

$$h(\eta) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log \left(\max \left(|\eta^{(i)}|, 1 \right) \right) \right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max(|p|, q)$. Several properties of the logarithmic height function $h(\eta)$, which will be used throughout the following sections without further mention, are also well established:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta \gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s|h(\eta) \quad (s \in \mathbb{Z}). \end{aligned}$$

In particular, if $\eta = p/q \in \mathbb{Q}$ is rational number in its reduced form with $q > 0$, then $h(\eta) = \log(\max\{|p|, |q|\})$.

Theorem 3.3. *Let \mathbb{K} be a number field of degree $d_{\mathbb{K}}$ over \mathbb{Q} , η_1, \dots, η_s be positive real numbers of \mathbb{K} , and b_1, \dots, b_s rational integers. Put*

$$\Lambda := \eta_1^{b_1} \dots \eta_s^{b_s}$$

and $B \geq \max(|b_1|, \dots, |b_s|)$.

Let $A_i \geq \max(d_{\mathbb{K}}h(\eta_i), |\log \eta_i|, 0.16)$ be real numbers for $i = 1, \dots, s$. With the above notation, Matveev [12] proved the following result. Then assuming that $\Lambda \neq 0$, we have

$$\Lambda > \exp(-1.4 \times 30^{s+3} \times s^{4.5} \times d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \dots A_s).$$

We also need the following result from Sánchez and Luca [13].

Lemma 3.4. (Lemma 7, [13])

If $l \geq 1, H > (4l^2)^l$ and $H > L/(\log L)^l$, then $L < 2^l H(\log H)^l$.

3.3. Reduction method. The bounds on the variables obtained via Baker’s theory [14] are too large for any computational purposes. To reduce the bounds we use the reduction method due to Dujella and Pethő [Lemma 5a, [15]].

Lemma 3.5. *Let M be a positive integer, p/q be a convergent of the continued fraction expansion of irrational number τ such as $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\|.$$

If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Here for a real number X , $\|X\|$ denotes the distance from X to the nearest integer, that is, $\|X\| := \min_{n \in \mathbb{Z}} |X - n|$.

4. PROOFS OF MAIN RESULT

4.1. Proof of Theorem 2.1.

Proof. In order to establish Theorem 2.1, we make use of the following lemma, which gives a bound relating the size of k to the parameters t and g .

Lemma 4.1. *All solutions of the Diophantine equation (2.1) satisfy*

$$k < 12t \log g.$$

Proof. To prove this lemma, we combine equation (2.1) with inequalities (3.1). Therefore, using $\ell \leq m \leq n \leq t$ and $1 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq g - 1$, we have:

$$\alpha^{k-2} \leq N_k = d_1 \frac{g^\ell - 1}{g - 1} \cdot d_2 \frac{g^m - 1}{g - 1} \cdot d_3 \frac{g^n - 1}{g - 1} \cdot d_4 \frac{g^t - 1}{g - 1} \leq (g^t - 1)^4 < g^{4t}.$$

Taking logarithm on the both sides, we get $(k - 2) \log \alpha < 4t \log g$.

Since $t \geq 2$ and $g \geq 2$, we obtain

$$k < \left(\frac{4}{\log \alpha} + \frac{2}{t \log g} \right) t \log g < 12t \log g. \text{ This ends the proof.}$$

□

Note that if $t = 1$, then $l = m = n = 1$. Thus the equation (2.1) becomes

$$N_k = d_1 d_2 d_3 d_4,$$

Which implies

$$\alpha^{k-2} \leq (g - 1)^4 < g^4.$$

And finally we have

$$k < 2 + 4 \frac{\log g}{\log \alpha} = \left(\frac{4}{\log \alpha} + \frac{2}{\log g} \right) \log g < 14 \log g \text{ since } g \geq 2.$$

Now, suppose that $t \geq 2$. From (2.1) and (3.2), we have

$$N_k = a\alpha^k + b\beta^k + c\gamma^k = d_1 \frac{g^\ell - 1}{g - 1} \cdot d_2 \frac{g^m - 1}{g - 1} \cdot d_3 \frac{g^n - 1}{g - 1} \cdot d_4 \frac{g^t - 1}{g - 1},$$

which implies

$$(4.1) \quad a\alpha^k - \frac{d_1 d_2 d_3 d_4 g^{\ell+m+n+t}}{(g-1)^4} = -\frac{d_1 d_2 d_3 d_4 (g^{\ell+m+n} + g^{\ell+m+t} + g^{\ell+n+t} + g^{m+n+t})}{(g-1)^4} + \frac{d_1 d_2 d_3 d_4 (g^{\ell+m} + g^{\ell+n} + g^{\ell+t} + g^{m+n} + g^{m+t} + g^{n+t})}{(g-1)^4} - \frac{d_1 d_2 d_3 d_4 (g^\ell + g^m + g^n + g^t)}{(g-1)^4} + \frac{d_1 d_2 d_3 d_4}{(g-1)^4} - \zeta_n.$$

Taking the absolute values of both sides of (4.1) and using (3.3), we get

$$(4.2) \quad \left| a\alpha^k - \frac{d_1 d_2 d_3 d_4 g^{\ell+m+n+t}}{(g-1)^4} \right| < \frac{d_1 d_2 d_3 d_4 (g^{\ell+m+n} + g^{\ell+m+t} + g^{\ell+n+t} + g^{m+n+t})}{(g-1)^4} + \frac{d_1 d_2 d_3 d_4 (g^{\ell+m} + g^{\ell+n} + g^{\ell+t} + g^{m+n} + g^{m+t} + g^{n+t})}{(g-1)^4} + \frac{d_1 d_2 d_3 d_4 (g^\ell + g^m + g^n + g^t)}{(g-1)^4} + \frac{d_1 d_2 d_3 d_4}{(g-1)^4} + \frac{1}{\alpha^{n/2}}.$$

When we multiply both sides of the above inequality by $\frac{(g-1)^4}{d_1 d_2 d_3 d_4 g^{\ell+m+n+t}}$ and using the fact $\ell \leq m \leq n \leq t$, we have

$$\begin{aligned} \left| \frac{(g-1)^4 a\alpha^k g^{-(\ell+m+n+t)}}{d_1 d_2 d_3 d_4} - 1 \right| &< \frac{1}{g^\ell} + \frac{1}{g^m} + \frac{1}{g^n} + \frac{1}{g^t} \\ &+ \frac{1}{g^{\ell+m}} + \frac{1}{g^{\ell+n}} + \frac{1}{g^{\ell+t}} + \frac{1}{g^{m+n}} + \frac{1}{g^{m+t}} + \frac{1}{g^{n+t}} \\ &+ \frac{1}{g^{\ell+m+n}} + \frac{1}{g^{\ell+m+t}} + \frac{1}{g^{\ell+n+t}} + \frac{1}{g^{m+n+t}} \\ &+ \frac{1}{g^{\ell+m+n+t}} + \frac{(g-1)^4}{d_1 d_2 d_3 d_4 g^{\ell+m+n+t} \alpha^{n/2}} \\ &< \frac{16}{g^\ell}. \end{aligned}$$

Which implies

$$(4.3) \quad \left| \frac{a(g-1)^4}{d_1 d_2 d_3 d_4} \cdot \alpha^k \cdot g^{-(\ell+m+n+t)} - 1 \right| < \frac{16}{g^\ell}.$$

So we can take

$$\Lambda_1 := \frac{a(g-1)^4}{d_1 d_2 d_3 d_4} \cdot \alpha^k \cdot g^{-(\ell+m+n+t)} - 1.$$

Let us show that $\Lambda_1 \neq 0$. We proceed by contrary. To that end, let us assume that $\Lambda_1 = 0$. Then

$$a\alpha = \frac{d_1 d_2 d_3 d_4}{(g-1)^4} \cdot g^{\ell+m+n+t},$$

which implies

$$\sigma_f(a\alpha^k) = b\beta^k = \frac{d_1 d_2 d_3 d_4}{(g-1)^4} \cdot g^{\ell+m+n+t}.$$

Taking the absolute value, we get

$$|b\beta^k| = \left| \frac{d_1 d_2 d_3 d_4}{(g-1)^4} \cdot g^{\ell+m+n+t} \right|.$$

We have $|b\beta^k| < 1$ instead $\left| \frac{d_1 d_2 d_3 d_4}{(g-1)^4} \cdot g^{\ell+m+n+t} \right| > 1$ since $1 \leq \ell \leq m \leq n \leq t$, which leads to a contradiction. Hence $\Lambda_1 \neq 0$.

With a view to applying Matveev's result to Λ_1 , let us set

$$s := 3, \eta_1 := \frac{a(g-1)^4}{d_1 d_2 d_3 d_4}, \eta_2 := \alpha, \eta_3 := g,$$

$$b_1 := 1, b_2 := k, b_3 := -(\ell + m + n + t),$$

and $\mathbb{K} = \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, which is a real number field of degree $d_{\mathbb{K}} = 3$.

Using the properties of the logarithmic height, it holds that,

$$h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3}, \quad h(\eta_3) = h(g) = \log g$$

and

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4}\right) \\ &\leq h(a) + h\left(\frac{(g-1)^4}{d_1 d_2 d_3 d_4}\right) \\ &\leq \frac{1}{3} \log 31 + \log(\max\{(g-1)^4, d_1 d_2 d_3 d_4\}) \\ &< 2 + 4 \log g = \left(4 + \frac{2}{\log g}\right) \log g < 7 \log g \text{ since } g \geq 2. \end{aligned}$$

Thus we can take

$$A_1 = 21 \log g, \quad A_2 := \log \alpha \text{ and } A_3 := 3 \log g.$$

Using Lemma 4.1 and $\ell \leq m \leq n \leq t$, we have

$$\max\{|b_1|, |b_2|, |b_3|\} = \max\{1, k, \ell + m + n + t\} < 12t \log g.$$

So we can put $B = 12t \log g$. Using Theorem 3.3, we obtain

$$\begin{aligned} \log |\Lambda_1| &> -1.4 \times 30^6 \times 3^{4.5} \times 3^2 \cdot (1 + \log 3) \cdot (1 + \log(12t \log g)) \\ &\quad \times (21 \log g)(\log \alpha)(3 \log g). \\ &> -6.5 \times 10^{13} (1 + \log(12t \log g)) (\log^2 g). \end{aligned}$$

Comparing the above inequality with (4.3) we obtain that

$$\ell \log g - \log 16 < 6.5 \times 10^{13}(1 + \log(12t \log g))(\log^2 g).$$

Since $g \geq 2$ and $t \geq 2$, we have

$$1 + \log(12t \log g) < 12 \log t \log g.$$

Hence we have

$$\ell < 7.8 \times 10^{14} \log t \log^2 g.$$

Using (2.1) and (3.1) once again and reorganizing the computations, we obtain

$$\frac{a\alpha^k(g-1)}{d_1(g^\ell-1)} + \frac{\zeta_k(g-1)}{d_1(g^\ell-1)} = \frac{d_2d_3d_4}{(g-1)^3}(g^{m+n+t} - g^{m+n} - g^{m+t} - g^{n+t} + g^m + g^n + g^t + 1).$$

Which implies

$$(4.4) \quad \begin{aligned} \frac{a\alpha^k(g-1)}{d_1(g^\ell-1)} - \frac{d_2d_3d_4g^{m+n+t}}{(g-1)^3} &= \frac{\zeta_k(g-1)}{d_1(g^\ell-1)} - \frac{d_2d_3d_4g^{m+n}}{(g-1)^3} \\ &- \frac{d_2d_3d_4g^{m+t}}{(g-1)^3} - \frac{d_2d_3d_4g^{n+t}}{(g-1)^3} + \frac{d_2d_3d_4g^m}{(g-1)^3} \\ &+ \frac{d_2d_3d_4g^n}{(g-1)^3} + \frac{d_2d_3d_4g^t}{(g-1)^3} + \frac{d_2d_3d_4}{(g-1)^3}. \end{aligned}$$

Taking the absolute value of the both sides of (4.4), we have

$$\begin{aligned} \left| \frac{a\alpha^k(g-1)}{d_1(g^\ell-1)} - \frac{d_2d_3d_4g^{m+n+t}}{(g-1)^3} \right| &< \frac{(g-1)}{d_1(g^\ell-1)\alpha^{k/2}} + \frac{d_2d_3d_4g^{m+n}}{(g-1)^3} + \frac{d_2d_3d_4g^{m+t}}{(g-1)^3} \\ &+ \frac{d_2d_3d_4g^{n+t}}{(g-1)^3} + \frac{d_2d_3d_4g^m}{(g-1)^3} + \frac{d_2d_3d_4g^n}{(g-1)^3} \\ &+ \frac{d_2d_3d_4g^t}{(g-1)^3} + \frac{d_2d_3d_4}{(g-1)^3}. \end{aligned}$$

Multiplying both sides of the inequality above by $\frac{(g-1)^3}{d_2d_3d_4g^{m+n+t}}$ and using the fact that $2 \leq m \leq n \leq t$, we get

$$\begin{aligned} \left| \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)} \cdot \alpha^k \cdot g^{-(m+n+t)} - 1 \right| &< \frac{(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)\alpha^{k/2}g^{m+n+t}} + \frac{1}{g^m} + \frac{1}{g^n} + \frac{1}{g^t} \\ &+ \frac{1}{g^{m+n}} + \frac{1}{g^{m+t}} + \frac{1}{g^{n+t}} + \frac{1}{g^{m+n+t}} < \frac{8}{g^m}. \end{aligned}$$

Then we have

$$(4.5) \quad \left| \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)} \cdot \alpha^k \cdot g^{-(m+n+t)} - 1 \right| < \frac{8}{g^m}.$$

Now, set

$$A_2 := \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)} \cdot \alpha^k \cdot g^{-(m+n+t)} - 1.$$

In a similar way, one can easily check that $\Lambda_2 \neq 0$, proceeding as we did for Λ_1 . Let us apply Matveev’s result for Λ_2 . Let

$$s := 3, \eta_1 := \frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)}, \eta_2 := \alpha, \eta_3 := g,$$

$$b_1 := 1, b_2 := k, b_3 := -(m + n + t)$$

and $\mathbb{K} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$ of degree $d_{\mathbb{K}} = 3$. By Lemma 4.1 we have $k < 12t \log g$, so we put $B := 12t \log g$. We have

$$h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3}, h(\eta_3) = h(g) = \log g$$

and

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)}\right) \\ &\leq h(a) + h\left(\frac{(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)}\right) \\ &\leq \frac{1}{3} \log 31 + \log(\max\{(g-1)^4, d_1 d_2 d_3 d_4\}) + h\left(\frac{1}{g^\ell - 1}\right) \\ &< 2 + 4 \log(g-1) + \log(g^\ell - 1) \\ &< (4 + \ell) \log g + 2 \\ &< (7 + \ell) \log g \text{ since } g \geq 2. \end{aligned}$$

Hence, since $\ell < 7.8 \times 10^{14} \log t \log^2 g$, we have

$$h(\eta_1) < 7.8 \times 10^{14} \log t \log^3 g.$$

Therefore we can take

$$A_1 := 2.34 \times 10^{15} \log t \log^3 g, A_2 := \log \alpha, \text{ and } A_3 := 3 \log g.$$

Using Theorem 3.3 we see that

$$\begin{aligned} \log |\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 3^2 (1 + \log 3) (1 + \log(12t \log g)) \\ &\quad \times (2.34 \times 10^{15} \log t \log^3 g) (\log \alpha) (3 \log g) \\ &> -7.26 \times 10^{27} (1 + \log(12t \log g)) \log t \log^4 g. \end{aligned}$$

Comparing with (4.5), we get

$$m \log g - \log 8 < 7.26 \times 10^{27} (1 + \log(12t \log g)) \log t \log^4 g.$$

And using the fact that

$$1 + \log(12t \log g) < 12 \log t \log g,$$

we have

$$m < 8.7 \times 10^{28} \log^2 t \log^4 g.$$

In the order to find an upper bound of n in terms of g and t we rewrite (2.1), using (3.1), so we get

$$\begin{aligned} \frac{a\alpha^k(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)} + \frac{\zeta_k(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)} &= \frac{d_3d_4(g^n-1)(g^t-1)}{(g-1)^2} \\ &= \frac{d_3d_4g^{n+t}}{(g-1)^2} - \frac{d_3d_4g^n}{(g-1)^2} \\ &\quad - \frac{d_3d_4g^t}{(g-1)^2} + \frac{d_3d_4}{(g-1)^2}. \end{aligned}$$

Then we have

$$(4.6) \quad \frac{a\alpha^k(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)} - \frac{d_3d_4g^{n+t}}{(g-1)^2} = -\frac{\zeta_k(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)} - \frac{d_3d_4g^n}{(g-1)^2} - \frac{d_3d_4g^t}{(g-1)^2} + \frac{d_3d_4}{(g-1)^2}.$$

Taking the absolute values of the both sides of (4.6) and using (3.3), we get

$$\begin{aligned} \left| \frac{a\alpha^k(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)} - \frac{d_3d_4g^{n+t}}{(g-1)^2} \right| &\leq \frac{(g-1)^2}{d_1d_2(g^\ell-1)(g^m-1)\alpha^{k/2}} \\ &\quad + \frac{d_3d_4g^n}{(g-1)^2} + \frac{d_3d_4g^t}{(g-1)^2} + \frac{d_3d_4}{(g-1)^2}. \end{aligned}$$

Multiplying both sides of inequality above by $\frac{(g-1)^2}{d_3d_4g^{n+t}}$ and noticing the fact that $2 \leq n \leq t$, we get,

$$\begin{aligned} \left| \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)(g^m-1)} \cdot \alpha^k \cdot g^{-(n+t)} - 1 \right| &< \frac{(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)(g^m-1)\alpha^{k/2}g^{n+t}} \\ &\quad + \frac{1}{g^n} + \frac{1}{g^t} + \frac{1}{g^{n+t}} \\ &< \frac{4}{g^n}. \end{aligned}$$

So we get

$$(4.7) \quad \left| \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)(g^m-1)} \cdot \alpha^k \cdot g^{-(n+t)} - 1 \right| < \frac{4}{g^n}.$$

We put

$$\Lambda_3 = \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)(g^m-1)} \cdot \alpha^k \cdot g^{-(n+t)} - 1.$$

One can verify that $\Lambda_3 \neq 0$. Let us apply once again Matveev's result for Λ_3 . Let

$$\begin{aligned} s := 3, \quad \eta_1 &= \frac{a(g-1)^4}{d_1d_2d_3d_4(g^\ell-1)(g^m-1)}, \quad \eta_2 := \alpha, \quad \text{and } \eta_3 := g, \\ b_1 &:= 1, \quad b_2 := k, \quad b_3 := -(n+t) \end{aligned}$$

and $\mathbb{K} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$ of degree $d_{\mathbb{K}} = 3$. By using Lemma 4.1 we have $k < 12t \log g$, so we can take $B = 12t \log g$. We have

$$h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3}, \quad h(\eta_3) = h(g) = \log g$$

and

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)}\right) \leq h(a) + h\left(\frac{(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)}\right) \\ &\leq \frac{1}{3} \log 31 + \log(\max\{(g-1)^4, d_1 d_2 d_3 d_4\}) + h\left(\frac{1}{g^\ell - 1}\right) + h\left(\frac{1}{g^m - 1}\right) \\ &< 2 + 4 \log(g-1) + \log(g^\ell - 1) + \log(g^m - 1) \\ &< (4 + \ell + m) \log g + 2 = \left(4 + \ell + m + \frac{2}{\log g}\right) \log g < (7 + \ell + m) \log g \\ &\quad \text{since } g \geq 2. \end{aligned}$$

Using the fact that $\ell < 7.8 \times 10^{14} \log t \log^2 g$ and $m < 8.7 \times 10^{28} \log^2 t \log^4 g$, we get

$$h(\eta_1) < 8.7 \times 10^{28} \log^2 t \log^5 g.$$

We take

$$A_1 := 2.61 \times 10^{28} \log^2 t \log^5 g, \quad A_2 := \log \alpha \quad \text{and} \quad A_3 = 3 \log g.$$

Using Theorem 3.3 we see that

$$\begin{aligned} \log |A_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 3^2 (1 + \log 3) (1 + \log(12t \log g)) \\ &\quad \times (2.61 \times 10^{28} \log^2 t \log^5 g) (3 \log g \log \alpha) \\ &> -8.1 \times 10^{41} (1 + \log(12t \log g)) \log^2 t \log^6 g. \end{aligned}$$

Comparing with (4.7), we get

$$n \log g - \log 4 < 8.1 \times 10^{41} (1 + \log(12t \log g)) \log^2 t \log^6 g.$$

We have

$$1 + \log(12t \log g) < 12 \log t \log g,$$

Thus

$$n < 9.72 \times 10^{42} \log^3 t \log^6 g.$$

This step will mark the end of the proof of Theorem 2.1. To get there we need to rearrange (2.1) in the following form in view to apply the Theorem 3.3:

$$\frac{a\alpha^k (g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)(g^n - 1)} + \frac{\zeta_k (g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)(g^n - 1)} = \frac{d_4 g^t}{(g-1)} - \frac{d_4}{(g-1)}.$$

Then we have

$$(4.8) \quad \frac{a\alpha^k (g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)(g^n - 1)} - \frac{d_4 g^t}{(g-1)} = -\frac{\zeta_k (g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)(g^n - 1)} - \frac{d_4}{(g-1)}.$$

Taking the absolute values of the both sides of (4.8) and using (3.3), we get

$$\left| \frac{a\alpha^k (g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)(g^n - 1)} - \frac{d_4 g^t}{(g-1)} \right| \leq \frac{(g-1)^3}{d_1 d_2 d_3 (g^\ell - 1)(g^m - 1)(g^n - 1) \alpha^{k/2}} + \frac{d_4}{(g-1)}.$$

Multiplying both sides of inequality above by $\frac{g-1}{d_4 g^t}$ and noticing the fact that $t \geq 2$, we get,

$$(4.9) \quad \left| \frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)} \alpha^k \cdot g^{-t} - 1 \right| < \frac{2}{g^{t-1}}.$$

We put

$$A_4 = \frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)} \cdot \alpha^k \cdot g^{-t} - 1.$$

One can verify that $A_4 \neq 0$. Let us analyze Matveev's result for A_4 . Let

$$s := 3, \quad \eta_1 = \frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)}, \quad \eta_2 := \alpha, \quad \text{and } \eta_3 := g,$$

$$b_1 := 1, \quad b_2 := k, \quad b_3 = -t$$

and $\mathbb{K} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$ of degree $d_{\mathbb{K}} = 3$. By using Lemma 4.1 we have $k < 12t \log g$, so we put $B = 12t \log g$. We have

$$h(\eta_2) = h(\alpha) = \frac{\log \alpha}{3}, \quad h(\eta_3) = h(g) = \log g$$

and

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)}\right) \\ &\leq h(a) + h\left(\frac{(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)}\right) \\ &\leq \frac{1}{3} \log 31 + \log(\max\{(g-1)^4, d_1 d_2 d_3 d_4\}) + h\left(\frac{1}{g^\ell - 1}\right) \\ &\quad + h\left(\frac{1}{g^m - 1}\right) + h\left(\frac{1}{g^n - 1}\right) \\ &< 2 + 4 \log(g-1) + \log(g^\ell - 1) + \log(g^m - 1) + \log(g^n - 1) \\ &< (4 + \ell + m + n) \log g + 2 \\ &= \left(4 + \ell + m + n + \frac{2}{\log g}\right) \log g \\ &< (7 + \ell + m + n) \log g, \quad \text{since } g \geq 2. \end{aligned}$$

Using the fact that

$$\ell < 7.8 \times 10^{14} \log t \log^2 g, \quad m < 8.7 \times 10^{28} \log^2 t \log^4 g \quad \text{and} \quad n < 9.72 \times 10^{42} \log^3 t \log^6 g,$$

we get

$$h(\eta_1) < 9.72 \times 10^{42} \log^3 t \log^7 g.$$

So we can take

$$A_1 := 2.92 \times 10^{43} \log^3 t \log^7 g, \quad A_2 := \log \alpha \quad \text{and} \quad A_3 := 3 \log g.$$

Using Theorem 3.3 we see that

$$\begin{aligned} \log |A_4| &> -1.4 \times 30^6 \times 3^{4.5} \times 3^2 (1 + \log 3) (1 + \log(12t \log g)) \\ &\quad \times (2.92 \times 10^{43} \log^3 t \log^7 g) (3 \log g \log \alpha) \\ &> -9.1 \times 10^{55} (1 + \log(12t \log g)) \log^3 t \log^8 g. \end{aligned}$$

Comparing with (4.9), we get

$$(t - 1) \log g - \log 2 < 9.1 \times 10^{55} (1 + \log(12t \log g)) \times \log^3 t \log^8 g.$$

We have

$$1 + \log(12t \log g) < 12 \log t \log g.$$

So we have

$$t < 1.1 \times 10^{57} \log^4 t \log^8 g.$$

To finally have the upper bound of t in term of g , we need to apply the Lemma 3.4 due to Sánchez and Luca.

Taking $l := 4$, $L := t$ and $H := 1.1 \times 10^{57} \log^8 g$, we get using Lemma above that

$$\begin{aligned} t &< 2^4 \times 1.1 \times 10^{57} \log^8 g \log^4 (1.1 \times 10^{57} \log^8 g) \\ &< 1.76 \times 10^{58} \log^8 g (131.35 + 8 \log \log g)^4 \\ t &< 2.11 \times 10^{67} \log^{12} g. \end{aligned}$$

Notice that we have use the inequality $131.35 + 8 \log \log g < 186 \log g$ which holds since $g \geq 2$.

Moreover, by Lemma 4.1, we get

$$k < 2.54 \times 10^{68} \log^{13} g.$$

□

4.2. Proof of Theorem 2.2.

Proof. In subsection 4.1, we established that for $2 \leq g \leq 12$,

$$\ell \leq m \leq n \leq t < 1.18 \times 10^{72} \text{ and } k < 3.5 \times 10^{73}.$$

The next objective is to refine the upper bounds above in order to delimit the interval containing the possible solutions of (2.1), following a four-step approach.

Step 1 Referring to (4.3), we introduce

$$\Gamma_1 := \log(\Lambda_1 + 1) = k \log \alpha - (l + m + n + t) \log g + \log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4} \right).$$

Notice that, since $\Gamma_1 = \log(\Lambda_1 + 1)$, we have $|e^{\Gamma_1} - 1| = |\Lambda_1| < \frac{16}{g^\ell}$.

Observe that $\Gamma_1 \neq 0$, since $\Lambda_1 \neq 0$. So, for $\ell \geq 5$ and $g \geq 2$, we have

$$|e^{\Gamma_1} - 1| < \frac{16}{g^\ell} < \frac{1}{2}.$$

Since $|x| < 2|e^x - 1|$, if $|x| < \frac{1}{2}$ holds, then

$$|\Gamma_1| < 2|e^{\Gamma_1} - 1| = 2|\Lambda_1| < \frac{32}{g^\ell}.$$

Substituting Γ_1 in the above inequality with its value and dividing through by $\log g$, we get

$$\left| k \left(\frac{\log \alpha}{\log g} \right) - (\ell + m + n + t) + \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4} \right)}{\log g} \right| < \frac{32}{\log g} \cdot g^{-\ell}.$$

Then, we can apply Lemma 3.5 with the data

$$\tau := \frac{\log \alpha}{\log g}, \quad \mu := \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4} \right)}{\log g}, \quad A := \frac{32}{\log g},$$

$$B := g, \quad w := \ell, \quad u = k, \quad \text{and } v = \ell + m + n + t,$$

with

$$1 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq g - 1.$$

We can take $M := 3.5 \times 10^{73}$, since $k < 12t \log g < 3.5 \times 10^{73}$. So, for the remaining proof, we use Mathematica to apply Lemma 3.5. If the first convergent q_t satisfies $q_t > 6M$ but does not meet the requirement $\varepsilon > 0$, we proceed to the next convergent and check it accordingly; however, this should not be understood as guaranteeing that such a q always exists.

Thus, we have the results given in Table 2:

TABLE 2. Upper bound on ℓ .

g	2	3	4	5	6	7
q_t	q_{153}	q_{141}	q_{147}	q_{145}	q_{134}	q_{151}
$\varepsilon \geq$	0.34	0.39	0.051	0.011	0.0037	0.000019
$\ell \leq$	257	162	130	112	101	96

g	8	9	10	11	12
q_t	q_{153}	q_{147}	q_{134}	q_{140}	q_{154}
$\varepsilon \geq$	0.0023	0.0016	0.006	0.00028	0.0062
$\ell \leq$	88	84	79	76	75

Therefore

$$1 \leq \ell \leq \frac{\log ((32/\log 2) \cdot q_{153}/0.34)}{\log 2} \leq 257.$$

Step 2 Now, we focus on locating the real range of m . To do this, let us consider

$$\Gamma_2 = \log(A_2 + 1) = k \log \alpha - (m + n + t) + \log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)} \right).$$

Thus inequality (4.5) becomes

$$|e^{\Gamma_2} - 1| < \frac{8}{g^m} < \frac{1}{2}.$$

Which holds for $m \geq 4$. Thus,

$$(4.10) \quad \left| k \left(\frac{\log \alpha}{\log g} \right) - (m + n + t) \log g + \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)} \right)}{\log g} \right| < \frac{16}{\log g} \cdot g^{-m}.$$

Therefore we can applied Lemma 3.5 to the above inequality (4.10) with the following data

$$\tau := \frac{\log \alpha}{\log g}, \quad \mu := \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)} \right)}{\log g}, \quad A := \frac{16}{\log g},$$

$$B := g, \quad w := m, \quad u = k, \quad \text{and } v = m + n + t$$

with

$$1 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq g - 1 \text{ and } 1 \leq \ell \leq 265.$$

We can take $M := 3.5 \times 10^{73}$, since $k < 12t \log g < 3.5 \times 10^{73}$.

With *Mathematica* we get the results given in Table 3:

TABLE 3. Upper bound on m .

g	2	3	4	5	6	7
q_t	q_{153}	q_{141}	q_{147}	q_{146}	q_{134}	q_{151}
$\varepsilon \geq$	0.0012	3×10^{-4}	3.3×10^{-4}	6.9×10^{-4}	6.6×10^{-4}	1.9×10^{-5}
$m \leq$	264	168	133	113	102	95

g	8	9	10	11	12
q_t	q_{153}	q_{147}	q_{134}	q_{140}	q_{154}
$\varepsilon \geq$	1.3×10^{-4}	2.7×10^{-5}	1.8×10^{-5}	1.1×10^{-5}	4.6×10^{-6}
$m \leq$	89	85	82	77	77

In all cases, we can conclude that

$$1 \leq m \leq \frac{\log((16/\log 2) \cdot q_{153}/0.0012)}{\log 2} \leq 265.$$

Step 3 For the third application of Lemma 3.5, using the inequality (4.7), we set

$$\Gamma_3 = \log(A_3 + 1) = k \log \alpha - (n + t) \log g + \log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)} \right).$$

Thus inequality (4.7) becomes

$$|e^{\Gamma_3} - 1| < \frac{4}{g^m} < \frac{1}{2}.$$

Which holds for $m \geq 3$. It follows that

$$(4.11) \quad \left| k \left(\frac{\log \alpha}{\log g} \right) - (n + t) + \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)} \right)}{\log g} \right| < \frac{8}{\log g} \cdot g^{-n}.$$

Since the conditions of Lemma 3.5 are satisfied, we proceed to apply this Lemma to the inequality (4.11) with the following data

$$\tau := \frac{\log \alpha}{\log g}, \quad \mu := \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)} \right)}{\log g}, \quad A := \frac{8}{\log g},$$

$$B := g, \quad w := m, \quad u = k, \quad \text{and } v = m + n + t$$

$$B := g, \quad w := n, \quad u = k, \quad \text{and } v = n + t$$

with

$$1 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq g - 1, \quad 1 \leq \ell \leq 257 \text{ and } 1 \leq m \leq 265.$$

We can take $M := 3.5 \times 10^{73}$, since $k < 12t \log g < 3.5 \times 10^{73}$.

With *Mathematica* we get the results given in Table 4:

TABLE 4. Upper bound on n .

g	2	3	4	5	6	7
q_t	q_{153}	q_{141}	q_{147}	q_{146}	q_{134}	q_{151}
$\varepsilon \geq$	4.28×10^{-6}	1.6×10^{-6}	3.39×10^{-6}	2.3×10^{-6}	2.1×10^{-6}	1.2×10^{-6}
$n \leq$	270	171	135	115	103	95

g	8	9	10	11	12
q_t	q_{153}	q_{147}	q_{134}	q_{140}	q_{154}
$\varepsilon \geq$	8×10^{-6}	3.2×10^{-7}	1.7×10^{-6}	7.3×10^{-7}	9×10^{-7}
$n \leq$	89	86	82	77	77

In all cases, we can conclude that

$$1 \leq n \leq \frac{\log((8/\log 2) \cdot q_{153}/4.28 \times 10^{-6})}{\log 2} \leq 271.$$

Step 4

Finally, to further reduce the bound on t , we set

$$\Gamma_4 = \log(A_3 + 1) = k \log \alpha - t \log g + \log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)} \right).$$

Therefore inequality (4.9) becomes

$$|e^{\Gamma_4} - 1| < \frac{2}{g^{t-1}} < \frac{1}{2}.$$

Which holds for $t \geq 3$. It follows that

$$(4.12) \quad \left| k \left(\frac{\log \alpha}{\log g} \right) - t + \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)} \right)}{\log g} \right| < \frac{4}{\log g} \cdot g^{-(t-1)}.$$

Since the conditions of Lemma 3.5 are satisfied, we proceed to apply this Lemma to the inequality (4.12) with the following data

$$\tau := \frac{\log \alpha}{\log g}, \quad \mu := \frac{\log \left(\frac{a(g-1)^4}{d_1 d_2 d_3 d_4 (g^\ell - 1)(g^m - 1)(g^n - 1)} \right)}{\log g}, \quad A := \frac{4}{\log g},$$

$$B := g, \quad w := t - 1, \quad u = k, \quad \text{and } v = t$$

with

$$1 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq g - 1, \quad 1 \leq \ell \leq 257,$$

$$1 \leq m \leq 265, \quad \text{and } 1 \leq n \leq 271.$$

We can take $M := 3.5 \times 10^{73}$, since $k < 12t \log g < 3.5 \times 10^{73}$.
 With *Mathematica* we get the results given in Table 5:

TABLE 5. Upper bound on t .

g	2	3	4	5	6	7
q_t	q_{153}	q_{141}	q_{147}	q_{145}	q_{134}	q_{151}
$\varepsilon \geq$	4.28×10^{-6}	1.6×10^{-6}	1.9×10^{-6}	2.3×10^{-6}	5.4×10^{-6}	6×10^{-6}
$t \leq$	269	170	135	115	103	94

g	8	9	10	11	12
q_t	q_{152}	q_{147}	q_{134}	q_{140}	q_{154}
$\varepsilon \geq$	8×10^{-6}	3.2×10^{-6}	9.6×10^{-6}	7.9×10^{-7}	2.8×10^{-6}
$t \leq$	88	85	80	77	76

In all cases, we can conclude that

$$1 \leq t \leq \frac{\log((2/\log 2) \cdot q_{153}/3.8 \times 10^{-7})}{\log 2} \leq 270$$

which is valid for all g such as $2 \leq g \leq 12$. In light of the above results, we need to check the equation (2.1) in the cases $2 \leq g \leq 12$ for $1 \leq d_1; d_2; d_3; d_4 \leq 11$, $1 \leq \ell \leq 257$, $1 \leq m \leq 265$, $1 \leq n \leq 271$, $1 \leq t \leq 270$ and $1 \leq k \leq 8051$. A quick inspection using *Mathematica* reveals that the Diophantine equation (2.1) in the range $2 \leq g \leq 12$ has only the solutions listed in the statement of Theorem 2.2. This completes the proof of Theorem 2.2. \square

REFERENCES

- [1] J. P. Allouche and T. Johnson, Narayana’s cows and delayed morphisms, J. d’Inform. Music. (1996) île de Tatihou, <https://hal.archives-ouvertes.fr/hal-02986050>.
- [2] G. Abou-Elela, A. Elsonbaty and M. Anwar, Two problems on Narayana numbers and repeated digit numbers, arXiv:2303.01718v2 [math.NT] (2023) 14 pp.
- [3] K. N. Adédji, A. Filipin and A. Togbé, Fibonacci and Lucas numbers as products of three repdigits in base g , Rend. Circ. Mat. Palermo (2023) 19 pp.
- [4] K. Bhoi, B. K. Patel and P. K. Ray, Narayana numbers as sums of two base b repdigits, Acta Comment. Univ. Tartu. Math. 26 (2022) 183–192.
- [5] P. Tiebekabe and K. N. Adédji, Narayana’s cows numbers which are concatenations of three repdigits in base g , arXiv:2304.00773v1 [math.NT] (2023) 15 pp.
- [6] K. N. Adédji, Balancing numbers which are products of three repdigits in base b . Bol. Soc. Mat. Mex. 29 (2023) 15 pp.
- [7] Y. Bugeau, M. Mignotte and S. Siksek, Classic and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, Ann. of Math. J. Math. 163 (2006) 969–1018.
- [8] K. Ben Yakkou, K. R. Kakanou and P. Tiebekabe, Narayana numbers as products of three repdigits in base g , Acta et Commentationes Universitatis Tartuensis de Mathematica 27 (2) (2023) 319–334. Available online at: <https://ojs.utlib.ee/index.php/ACUTM>.
- [9] N. J. A. Sloane and OEIS Foundation Inc., The On-Line Encyclopedia of Integers Sequence, published electronically at <https://oeis.org/>, 2023.
- [10] K. N. Adédji, On Mulatu and generalized Lucas numbers which are products of four b -repdigits with a consequence, Indian J Pure Appl Math (2025). <https://doi.org/10.1007/s13226-025-00758-w>.

- [11] J. L. Ramírez and V. F. Sirvent, A note on the k-narayana sequence, *Ann. Math. Inform.* 45 (2015) 91–105.
- [12] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II, *Izv. Math.* 64 (2000) 1217–1269.
- [13] S. G. Sánchez and F. Luca, Linear combinations of factorials and S-units in a binary recurrence sequence, *Ann. Math. Qué.* 38 (2014) 169–188.
- [14] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Q. J. Math.* 20 (1969) 129–137.
- [15] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Q. J. Math.* 49 (1998) 291–306.

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